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The Cameron–Martin–Wiener method in turbulence and in Burgers' model: general formulae, and application to late decay

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We apply the Cameron–Martin–Wiener (formerly 'Wiener–Hermite') expansion of a random velocity field to the analytical study of turbulence. The kernels of this expansion contain all statistical information about the ensemble. Complete expressions are derived for constructing statistical quantities in terms of the kernels, and for the equations of motion of the kernels. We rigorously prove the Gaussian trend of the velocity field of the Navier–Stokes equation in the very late stage when the non-linear term is neglected. The n-dependence (n is the order of derivative) of the flatness factor, minus three for derivatives of the velocity field, shows a rapid increase with n in this stage.

The late decay problem of the Burgers model of turbulence is studied analytically with a view to obtaining suggestive guidelines for fitting the non-linear aspects of the model turbulence. We can divide the energy spectrum density into two parts, the larger of which is a kind of steady solution, which we call the 'equilibrium state', which remains self-similar in time in terms of an appropriate variable. The deviation from this 'equilibrium solution' satisfies the Kármán–Howarth equation. As initial velocity field, we take two particular cases: (*a*) a pure Gaussian, and (*b*) a non-Gaussian velocity field. With these two cases a detailed spectral analysis has been obtained. The energy spectrum deviation from 'equilibrium' declines exponentially to zero for all wave-numbers. The Gaussian case shows that the flatness factor minus three increases rapidly with n, while the non-Gaussian case does not show any marked dependence on n.

1. Introduction

Cameron & Martin (1947) and Wiener (1958) proposed a new method of investigating a non-linear random process. This is to expand a random process in an infinite series, in which the first term is an exact Gaussian process and the higher order terms contribute successive corrections to the Gaussian form, while each of the terms is statistically orthogonal to every other one. This idea has been

developed and formulated in detail by Imamura, Meecham & Siegel (1965). We call this expansion the Cameron–Martin–Wiener† expansion.

Even though the Cameron–Martin–Wiener expansion is quite general for any random process, it seems more likely to be useful when the process is nearly Gaussian as in turbulence (Simmons & Salter 1934; Townsend 1947; Stewart 1951; Batchelor & Townsend 1949; Frenkiel & Klebanoff 1967), since it is then possible to suppose that the convergence of the series can be made rapid. In view of these advantages, a number of studies (Meecham & Siegel 1964; Siegel, Imamura & Meecham 1965; Meecham & Jeng 1968) has been carried out for problems of turbulence and of the Burgers model of turbulence.

In the present paper, it will be shown how any statistical quantities can be explicitly expressed by the kernels of the Cameron–Martin–Wiener expansion, so that the problem of studying the random processes is reduced to finding a set of non-random, ordinary functions, the Cameron–Martin–Wiener kernels. This method is also applied to the rigorous proof of the Gaussian trend (Batchelor 1953) of the velocity field in the very late decay state of turbulence and to the analysis of Burgers' model of turbulence in the late stage.

2. The Cameron–Martin–Wiener expansion

The Cameron–Martin–Wiener expansion is based on the ideal random function. Let $a_\alpha(\mathbf{x})$ be the α-component of the vector function $\mathbf{a}(\mathbf{x})$. The ideal random function has the properties,

$$\left.\begin{aligned} \langle a_\alpha(\mathbf{x})\rangle &= 0, \\ \langle a_{\alpha_1}(\mathbf{x}_1)a_{\alpha_2}(\mathbf{x}_2)\rangle &= \delta_{\alpha_1,\alpha_2}\delta(\mathbf{x}_1-\mathbf{x}_2), \end{aligned}\right\} \tag{2.1}$$

plus further moment equations expressing the condition that $a_\alpha(\mathbf{x})$ be Gaussian (Wang & Uhlenbeck 1945). Consider a set of functionals,

$$H^{(1)}_{\alpha_1}(\mathbf{x}_1), \quad H^{(2)}_{\alpha_1\alpha_2}(\mathbf{x}_1,\mathbf{x}_2), \ldots .$$

These functionals are constructed in terms of the ideal random functions as follows:

$$\begin{aligned} H^{(m)}_{\alpha_1\ldots\alpha_m}(\mathbf{x}_1,\ldots,\mathbf{x}_m) &= \prod_{i=1}^{m} a_{\alpha_i}(\mathbf{x}_i) - \sum_{i>l=1}^{m}\sum_{\substack{i=1\\ i\neq j,\,l}}^{m} a_{\alpha_i}(\mathbf{x}_i)\,\delta_{\alpha_j,\alpha_l}\delta(\mathbf{x}_j-\mathbf{x}_l) \\ &\quad + \sum_{p>q=1}^{m}\sum_{\substack{j>l=1\\ j,\,l\neq p,\,q}}^{m}\prod_{\substack{i=1\\ i\neq p,\,q\\ i\neq j,\,l}}^{m} a_{\alpha_i}(\mathbf{x}_i)\,\delta_{\alpha_j,\alpha_l}\delta(\mathbf{x}_j-\mathbf{x}_l)\,\delta_{\alpha_p,\alpha_q}\delta(\mathbf{x}_p-\mathbf{x}_q) + \ldots . \end{aligned} \tag{2.2}$$

The functionals defined above are symmetric with respect to permutations of the arguments, $(\alpha_1,\mathbf{x}_1)$, $(\alpha_2,\mathbf{x}_2)$, ... From the definition of the functionals (2.2)

† The terminology 'Wiener–Hermite', originally proposed by Meecham & Siegel (1964), is not in accordance with scientific precedent because it was not Wiener, but Cameron & Martin (1947) who first proposed these functionals in the scientific literature. On the other hand, they are so completely identified with Wiener's life work that it seems highly appropriate to include his name in any name for the functionals. As a way of recognizing the role of all concerned, we therefore propose the name 'Cameron–Martin–Wiener'.

and the properties of ideal random function (2.1) follows the general relation of orthogonality (Imamura *et al.* 1965):

$$\langle H^{(n_1)}_{\alpha(1),\ldots,\alpha(n_1)}(\mathbf{x}_1,\ldots,\mathbf{x}_{n_1})\, H^{(n_2)}_{\alpha(n_1+1),\ldots,\alpha(n_1+n_2)}(\mathbf{x}_{n_1+1},\ldots,\mathbf{x}_{n_1+n_2}) \ldots H^{(n_l)}_{\alpha(n_1+\ldots+n_{l-1}+1),\ldots,\alpha(n_1+\ldots+n_l)}(\mathbf{x}_{n_1+\ldots+n_{l-1}+1},\ldots,\mathbf{x}_{n_1+\ldots+n_l})\rangle = \begin{cases} \sum_{\substack{\text{distinct}\\ \text{exogamous}\\ \text{pairings}}} \prod_{\substack{\text{exogamous}\\ \text{pairs}}} \delta_{\alpha_i,\alpha_j}\delta(\mathbf{x}_i-\mathbf{x}_j), \\ 0 \text{ otherwise.} \end{cases} \tag{2.3}$$

Any expectation value of products of H's which cannot be reduced to the form of the right-hand side of the above relation will vanish. Hence, if $(n_1+\ldots+n_l)$ is odd, the expectation value must vanish. The simplest case of the above relation is

$$\langle H^{(n)}_{\alpha_1\ldots\alpha_n}(\mathbf{x}_1,\ldots,\mathbf{x}_m)\, H^{(m)}_{\beta_1,\ldots,\beta_m}(\mathbf{y}_1,\ldots,\mathbf{y}_m)\rangle = \delta_{n,m}\Sigma\delta_{\alpha_1,\beta_{i(1)}}\delta(\mathbf{x}_1-\mathbf{y}_{i(1)})\ldots\delta_{\alpha_n,\beta_{i(n)}}\delta(\mathbf{x}_n-\mathbf{y}_{i(n)}), \tag{2.4}$$

where Σ means the sum over all possible permutations of the set of numbers $(i(1),\ldots,i(m))$.

An arbitrary random vector in three-dimensional space may be expanded in terms of the statistically orthogonal set of functionals defined in (2.2). In particular, an arbitrary random function $f_\alpha(\mathbf{x};t)$ with vanishing mean value and homogeneous in $\mathbf{x}$ space is expressed as follows:

$$f_\alpha(\mathbf{x};t) = \sum_{m=1}^{\infty}\int\ldots\int K^{(m)}_{\alpha_1\ldots\alpha_m}(\mathbf{x}-\boldsymbol{\eta}_1,\ldots,\mathbf{x}-\boldsymbol{\eta}_m;t) \times H^{(m)}_{\alpha_1\ldots\alpha_m}(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_m)\, d\boldsymbol{\eta}_1\ldots d\boldsymbol{\eta}_m, \tag{2.5}$$

where the summation convention for the repeated indices $\alpha_1,\ldots,\alpha_m$ is understood. This is the Cameron–Martin–Wiener expansion. The functionals $H^{(m)}$ which are the basis of the expansion may be called Cameron–Martin–Wiener functionals. The non-random function $K^{(m)}$, the Cameron–Martin–Wiener kernel, is symmetric with respect to permutation of its arguments, $(\alpha_1,\mathbf{x}_1),\ldots,(\alpha_m,\mathbf{x}_m)$, and is a tensor of rank $m+1$. The absence of the constant term in the expression implies the vanishing mean value of the random function. The homogeneity of the field is guaranteed by taking the Cameron–Martin–Wiener kernels to be functions of difference arguments only.

As with ordinary non-random functions, the Fourier transform of the random function can be defined

$$\tilde{a}_\alpha(\boldsymbol{\xi}) = \int a_\alpha(\boldsymbol{\eta})\exp(i\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\boldsymbol{\eta}. \tag{2.6}$$

The fundamental properties of $\tilde{a}_\alpha(\boldsymbol{\xi})$ are

$$\langle\tilde{a}_\alpha(\boldsymbol{\xi})\rangle = \int\langle a_\alpha(\boldsymbol{\eta})\rangle\exp(i\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, d\boldsymbol{\eta} = 0,$$

$$\langle\tilde{a}_{\alpha_1}(\boldsymbol{\xi}_1)\,\tilde{a}_{\alpha_2}(\boldsymbol{\xi}_2)\rangle = \iint\langle a_{\alpha_1}(\boldsymbol{\eta}_1)\, a_{\alpha_2}(\boldsymbol{\eta}_2)\rangle\exp\{i(\boldsymbol{\xi}_1\boldsymbol{\eta}_1+\boldsymbol{\xi}_2\boldsymbol{\eta}_2)\}d\boldsymbol{\eta}_1 d\boldsymbol{\eta}_2 = (2\pi)^3\delta_{\alpha_1,\alpha_2}\delta(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2). \tag{2.7}$$

If we write the Fourier transform of the Cameron–Martin–Wiener functionals as

$$\tilde{H}^{(m)}_{\alpha_1\ldots\alpha_m}(\xi_1,\ldots,\xi_m) = \int\ldots\int H^{(m)}_{\alpha_1\ldots\alpha_m}(\eta_1,\ldots,\eta_m) \times \exp\{i(\xi_1\cdot\eta_1+\ldots+\xi_m\cdot\eta_m)\}\,d\eta_1\ldots d\eta_m,$$

the Fourier transform of the general orthogonality relation (2.3) reduces to

$$\langle \tilde{H}^{(n_1)}_{\alpha(1),\ldots,\alpha(n_1)}(\xi_1,\ldots,\xi_{n_1})\,\tilde{H}^{(n_2)}_{\alpha(n_1+1),\ldots,\alpha(n_1+n_2)}(\xi_{n_1+1},\ldots,\xi_{n_1+n_2}) \ldots \tilde{H}^{(n_l)}_{\alpha(n_1+\ldots+n_{l-1}+1),\ldots,\alpha(n_1+\ldots+n_l)}(\xi_{n_1+\ldots+n_{l-1}+1},\ldots,\xi_{n_1+\ldots+n_l})\rangle = \begin{cases} (2\pi)^{\frac{3}{2}(n_1+\ldots+n_l)} \sum\limits_{\substack{\text{distinct}\\ \text{exogamous}\\ \text{pairings}}} \prod\limits_{\substack{\text{exogamous}\\ \text{pairs}}} \delta_{\alpha i,\alpha j}\,\delta(\xi_i+\xi_j), \\ 0 \text{ otherwise.} \end{cases} \tag{2.8}$$

In the homogeneous field, the Fourier transform of a random function with vanishing mean value is expanded as follows

$$\tilde{f}_\alpha(\mathbf{k},t) = \sum_{m=1}^{\infty}\left(\frac{1}{2\pi}\right)^{3(m-1)}\int\ldots\int d\mathbf{k}_1\ldots d\mathbf{k}_m\,\tilde{K}^{(m)}_{\alpha\alpha_1\ldots\alpha_m}(\mathbf{k}_1,\ldots,\mathbf{k}_m;t) \times \tilde{H}_{\alpha_1\ldots\alpha_m}(\mathbf{k}_1,\ldots,\mathbf{k}_m)\,\delta(\mathbf{k}-\mathbf{k}_1-\ldots-\mathbf{k}_m). \tag{2.9}$$

It is easily seen that the delta function, $\delta(\mathbf{k}-\mathbf{k}_1-\ldots-\mathbf{k}_m)$, guarantees the homogeneity of the field.

3. Expressions for the velocity moments

We shall assume the field to be such that every statistical quantity is expressible in terms of the velocity moments. In the following, we shall show explicitly how to construct expressions for the velocity moments in terms of the Cameron–Martin–Wiener kernels.

In this section, the discussion is primarily restricted to the three-dimensional field. The expressions for the one-dimensional field, such as that of Burgers' model, are particular cases of the corresponding expressions for the three-dimensional field.

3.1. *The velocity correlation tensor and the energy spectrum tensor*

The velocity correlation tensor of the homogeneous velocity field is defined by

$$Q_{ij}(\mathbf{r},t) \equiv \langle u_i(\mathbf{x},t)\,u_j(\mathbf{x}+\mathbf{r};t)\rangle, \tag{3.1}$$

whose Fourier transform is the energy spectrum tensor,

$$\tilde{Q}_{ij}(\mathbf{k},t) = \int Q_{ij}(\mathbf{r},t)\exp(i\mathbf{k}\cdot\mathbf{r})\,d\mathbf{r}. \tag{3.2}$$

The expression for these two tensors in terms of Cameron–Martin–Wiener kernels is obtained by substituting the expansion of (2.5) and applying the orthogonality of relation (2.4): the correlation tensor is given by

$$Q_{ij}(\mathbf{r},t) = \sum_{m=1}^{\infty} m!\int\ldots\int K^{(m)}_{i\alpha_1\ldots\alpha_m}(\mathbf{x}_1,\ldots,\mathbf{x}_m;t) \times K^{(m)}_{j\alpha_1\ldots\alpha_m}(\mathbf{x}_1+\mathbf{r},\ldots,\mathbf{x}_m+\mathbf{r};t)\,d\mathbf{x}_1\ldots d\mathbf{x}_m, \tag{3.3}$$

and thus the energy spectrum tensor is

$$\tilde{Q}_{ij}(\mathbf{k},t) = \sum_{m=1}^{\infty} \frac{m!}{(2\pi)^{3(m-1)}} \int \dots \int d\mathbf{k}_1 \dots d\mathbf{k}_m \tilde{K}^{(m)}_{i\alpha_1\dots\alpha_m}(\mathbf{k}_1, \dots, \mathbf{k}_m; t) \times \tilde{K}^{(m)}_{j\alpha_1\dots\alpha_m}(-\mathbf{k}_1, \dots, -\mathbf{k}_m; t)\, \delta(\mathbf{k}+\mathbf{k}_1+\dots+\mathbf{k}_m). \quad (3.4)$$

3.2. *A triple velocity correlation tensor*

One of the most important cases of a triple velocity correlation tensor in the homogeneous field is the two-point triple correlation tensor defined by

$$S_{ij,l}(\mathbf{r},t) = \langle u_i(\mathbf{x},t)\, u_j(\mathbf{x},t)\, u_l(\mathbf{x}+\mathbf{r};t)\rangle. \quad (3.5)$$

We shall give the evaluation of this quantity in terms of Cameron–Martin–Wiener kernels in some detail both because of its intrinsic interest and because the methods used are typical.

Substituting the Cameron–Martin–Wiener expansion of the velocity field and dropping the identically vanishing terms according to the orthogonality relation, the triple correlation tensor is written

$$\begin{aligned} S_{ij,l}(\mathbf{r},t) = {\sum_{p,q,s}}' \int \dots \int & d\boldsymbol{\eta}_1 \dots d\boldsymbol{\eta}_{p+q} d\boldsymbol{\zeta}_1 \dots d\boldsymbol{\zeta}_{p+s} d\boldsymbol{\xi}_1 \dots d\boldsymbol{\xi}_{q+s} \\ & \times K^{(p+q)}_{i\alpha_1\dots\alpha_{p+q}}(\mathbf{x}-\boldsymbol{\eta}_1, \dots, \mathbf{x}-\boldsymbol{\eta}_p, \mathbf{x}-\boldsymbol{\eta}_{p+1}, \dots, \mathbf{x}-\boldsymbol{\eta}_{p+q}; t) \\ & \times K^{(p+s)}_{j\beta_1\dots\beta_{p+s}}(\mathbf{x}-\boldsymbol{\zeta}_1, \dots, \mathbf{x}-\boldsymbol{\zeta}_p, \mathbf{x}-\boldsymbol{\zeta}_{p+1}, \dots, \mathbf{x}-\boldsymbol{\zeta}_{p+s}; t) \\ & \times K^{(q+s)}_{l\gamma_1\dots\gamma_{q+s}}(\mathbf{x}+\mathbf{r}-\boldsymbol{\xi}_1, \dots, \mathbf{x}+\mathbf{r}-\boldsymbol{\xi}_q, \mathbf{x}+\mathbf{r}-\boldsymbol{\xi}_{q+1}, \dots, \mathbf{x}+\mathbf{r}-\boldsymbol{\xi}_{q+s}; t) \\ & \times \langle H^{(p+q)}_{\alpha_1\dots\alpha_{p+q}}(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_p, \boldsymbol{\eta}_{p+1}, \dots, \boldsymbol{\eta}_{p+q}) \\ & \times H^{(p+s)}_{\beta_1\dots\beta_{p+s}}(\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_p, \boldsymbol{\zeta}_{p+1}, \dots, \boldsymbol{\zeta}_{p+s}) \\ & \times H^{(q+s)}_{\gamma_1\dots\gamma_{q+s}}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_q, \boldsymbol{\xi}_{q+1}, \dots, \boldsymbol{\xi}_{q+s})\rangle, \end{aligned} \quad (3.6)$$

where the summation runs from zero to infinity over p, q and s with the restrictions

$$(p+q) > 0, \quad (p+s) > 0, \quad (q+s) > 0. \quad (3.7)$$

According to the orthogonality relation, the evaluation of the expectation value which is shown in the integral of (3.6) can be made in the following steps: p variables and p indices of ($\boldsymbol{\eta}$'s, α's) and of ($\boldsymbol{\xi}$'s, β's) make couples giving a product of p delta functions and of p Kronecker deltas. (A particular case is, for example,

$$\delta(\boldsymbol{\eta}_1-\boldsymbol{\zeta}_1)\, \delta_{\alpha_1,\beta_1} \dots \delta(\boldsymbol{\eta}_p-\boldsymbol{\zeta}_p)\, \delta_{\alpha_p,\beta_p}, \quad (3.8)$$

where the variable $\boldsymbol{\eta}_i$ and the index α_i always go together and similarly the variable $\boldsymbol{\zeta}_j$ and index β_j.) The total number of ways of making products similar to that given by (3.8) is

$$\binom{p+q}{p}\binom{p+s}{p} p! = \frac{(p+q)!\,(p+s)!}{p!\,q!\,s!}. \quad (3.9)$$

In addition to the above-mentioned type of product, there are other products which must finally be multiplied by the product (3.8). The others are the remaining q variables and q indices of ($\boldsymbol{\eta}$'s, α's) and the q variables and q indices of ($\boldsymbol{\xi}$'s, γ's), which make couples giving the following type of product,

$$\delta(\boldsymbol{\eta}_{p+1}-\boldsymbol{\xi}_1)\, \delta_{\alpha_{p+1},\gamma_1} \dots \delta(\boldsymbol{\eta}_{p+q}-\boldsymbol{\xi}_q)\, \delta_{\alpha_{p+q},\gamma_q}, \quad (3.10)$$

while for the remaining s variables and s indices of (ζ's, β's) and of (ξ's, γ's) the following type of product is obtained:

$$\delta(\zeta_{p+1}-\xi_{q+1})\,\delta_{\beta_{p+1},\gamma_{q+1}}\cdots\delta(\zeta_{p+s}-\xi_{q+s})\,\delta_{\beta_{p+s},\gamma_{q+s}}. \tag{3.11}$$

The total number of ways of making products similar to those of (3.10) and (3.11) are respectively

$$\binom{q+s}{q}\cdot q! = \frac{(q+s)!}{s!} \tag{3.12}$$

and

$$s!. \tag{3.13}$$

When we substitute these products for the evaluation of the expectation value into (3.6), we easily see that each product gives exactly the same result. Thus the sum of all such products can be replaced by one particular product, namely (3.8) times (3.10) times (3.11), multiplied by the numbers given by (3.9), (3.12) and (3.13). We then get

$$\begin{aligned} S_{ij,l}(\mathbf{r},t) = {\sum_{p,q,s}}' \frac{(p+q)!\,(p+s)!\,(q+s)!}{p!\,q!\,s!} \\ \times \int\cdots\int d\mathbf{x}_1\cdots d\mathbf{x}_p\,d\mathbf{x}_{p+1}\cdots d\mathbf{x}_{p+q}\,d\mathbf{x}_{p+q+1}\cdots d\mathbf{x}_{p+q+s} \\ \times K^{(p+q)}_{i\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_{p+q}}(\mathbf{x}_1,\ldots,\mathbf{x}_p,\mathbf{x}_{p+1},\ldots,\mathbf{x}_{p+q};t) \\ \times K^{(p+s)}_{j\alpha_1\ldots\alpha_p\alpha_{p+q+1}\ldots\alpha_{p+q+s}}(\mathbf{x}_1,\ldots,\mathbf{x}_p,\mathbf{x}_{p+q+1},\ldots,\mathbf{x}_{p+q+s};t) \\ \times K^{(p+s)}_{l\alpha_{p+1}\ldots\alpha_{p+q}\alpha_{p+q+1}\ldots\alpha_{p+q+s}}(\mathbf{x}_{p+1}+\mathbf{r},\ldots,\mathbf{x}_{p+q}+\mathbf{r}, \\ \mathbf{x}_{p+q+1}+\mathbf{r},\ldots,\mathbf{x}_{p+q+s}+\mathbf{r};t). \end{aligned} \tag{3.14}$$

3.3. *Fourth- and higher-order velocity moments*

Consider the following one-point fourth-order velocity moment,

$$\langle u_i(\mathbf{x},t)\,u_j(\mathbf{x},t)\,u_l(\mathbf{x},t)\,u_m(\mathbf{x},t)\rangle. \tag{3.15}$$

Instead of expressing this directly in terms of the Cameron–Martin–Wiener kernels, let us introduce the fourth-order cumulant. The fourth-order cumulant is defined as the fourth-order moment minus all possible combinations of two products of second-order moments, i.e., if we write for the fourth-order cumulant $N_4(i,j,l,m;t)$, then

$$\begin{aligned} N_4(i,j,l,m;t) = {} & \langle u_i(\mathbf{x},t)\,u_j(\mathbf{x},t)\,u_l(\mathbf{x},t)\,u_m(\mathbf{x},t)\rangle \\ & - \langle u_i(\mathbf{x},t)\,u_j(\mathbf{x},t)\rangle\,\langle u_l(\mathbf{x},t)\,u_m(\mathbf{x},t)\rangle \\ & - \langle u_i(\mathbf{x},t)\,u_l(\mathbf{x},t)\rangle\,\langle u_j(\mathbf{x},t)\,u_m(\mathbf{x},t)\rangle \\ & - \langle u_i(\mathbf{x},t)\,u_m(\mathbf{x},t)\rangle\,\langle (u_j(\mathbf{x},t)\,u_l(\mathbf{x},t)\rangle. \end{aligned} \tag{3.16}$$

By definition, the fourth-order cumulant is not expressible in terms of the product of two, or more than two, lower-order velocity moments. Hence the expression of the fourth-order cumulant through the Cameron–Martin–Wiener kernels should contain four kernels in each of the terms, but no term can be expressed

as the product of two or more than two integrals. Following the arguments and using notations similar to those of the third-order moment, we obtain

$$
\begin{aligned}
&N_4(i,j,l,m;t) \\
&= \sum_{a,b,c,d,e,f}{}' \frac{(a+b+c)!\,(a+d+e)!\,(b+d+f)!\,(c+e+f)!}{a!\,b!\,c!\,d!\,e!\,f!} \\
&\quad \times \int \cdots \int d\mathbf{x}_1^{(1)} \cdots d\mathbf{x}_a^{(1)} d\mathbf{x}_1^{(2)} \cdots d\mathbf{x}_b^{(2)} \cdots d\mathbf{x}_c^{(3)} \cdots d\mathbf{x}_d^{(4)} \cdots d\mathbf{x}_e^{(5)} \cdots d\mathbf{x}_f^{(6)} \\
&\quad \times K^{(a+b+c)}_{i\alpha_1(1)\cdots\alpha_a(1)\,\alpha_1(2)\cdots\alpha_b(2)\,\alpha_1(3)\cdots\alpha_c(3)}(\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_a^{(1)}, \mathbf{x}_1^{(2)}, \ldots, \mathbf{x}_b^{(2)}, \mathbf{x}_1^{(3)}, \ldots, \mathbf{x}_c^{(3)}; t) \\
&\quad \times K^{(a+d+e)}_{j\alpha_1(1)\cdots\alpha_a(1)\,\alpha_1(4)\cdots\alpha_d(4)\,\alpha_1(5)\cdots\alpha_e(5)}(\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_a^{(1)}, \mathbf{x}_1^{(4)}, \ldots, \mathbf{x}_d^{(4)}, \mathbf{x}_1^{(5)}, \ldots, \mathbf{x}_e^{(5)}; t) \\
&\quad \times K^{(b+d+f)}_{l\alpha_1(2)\cdots\alpha_b(2)\,\alpha_1(4)\cdots\alpha_d(4)\,\alpha_1(6)\cdots\alpha_f(6)}(\mathbf{x}_1^{(2)}, \ldots, \mathbf{x}_b^{(2)}, \mathbf{x}_1^{(4)}, \ldots, \mathbf{x}_d^{(4)}, \mathbf{x}_1^{(6)}, \ldots, \mathbf{x}_f^{(6)}; t) \\
&\quad \times K^{(c+e+f)}_{m\alpha_1(3)\cdots\alpha_c(3)\,\alpha_1(5)\cdots\alpha_e(5)\,\alpha_1(6)\cdots\alpha_f(6)}(\mathbf{x}_1^{(3)}, \ldots, \mathbf{x}_c^{(3)}, \mathbf{x}_1^{(5)}, \ldots, \mathbf{x}_e^{(5)}, \mathbf{x}_1^{(6)}, \ldots, \mathbf{x}_f^{(6)}; t),
\end{aligned}
\tag{3.17}
$$

where a, b, c, d, e and f run from zero to infinity, with the following restrictions

$$(a+b+c) > 0, \quad (a+d+e) > 0, \quad (b+d+f) > 0, \quad (c+e+f) > 0.$$

The computation of the rth order moment would be quite similar. It is most readily expressed in terms of the rth order cumulant, defined as the rth order velocity moment minus all possible terms which are expressed as products of the lower-order moments and/or lower-order cumulants. The next step is to express the rth order cumulant in Cameron–Martin–Wiener series. In this expression, the most important thing is the way of assigning the notations, i.e. the variables are divided into $(\frac{1}{2})r(r-1)$ sets, each of which is distinguished by an attached number as a superscript of the variable $\mathbf{x}$.

The property expressed by (3.17) is extremely important in understanding the way in which the Cameron–Martin–Wiener kernels express the statistical properties of the ensemble; the simplicity of this result compared to that which would be obtained for the moments shows that the Cameron–Martin–Wiener kernels fit in a particularly natural way with the expression of the statistical properties in terms of *cumulants* rather than *moments*.

4. Derivation of the equations of motion of the kernels

The time dependence of the Cameron–Martin–Wiener kernels is obtained essentially from the Navier–Stokes equation. Substituting the Cameron–Martin–Wiener expansion of the velocity field into the Navier–Stokes equation and utilizing the statistical orthogonality of the Cameron–Martin–Wiener functionals, one obtains an infinite number of integro-differential equations. The solution of these equations would express the statistical quantities of turbulence as functions of time.

Siegel, Imamura & Meecham (1965) derived the equations of motion of the Cameron–Martin–Wiener kernels for the Burgers equation, while Meecham &

Jeng (1968) did the same for the Navier–Stokes equation; both results, however, were with neglect of higher-order kernels. We shall give here a completely general result, including all kernels.

The incompressibility of the fluid gives

$$\frac{\partial}{\partial x_i} u_i(\mathbf{x}, t) = 0. \tag{4.1}$$

Substituting the Cameron–Martin–Wiener expansion of the velocity field and applying the orthogonality of the functionals, this becomes

$$\frac{\partial}{\partial x_i} K^{(m)}_{i\alpha_1 \ldots \alpha_m}(\mathbf{x}-\boldsymbol{\eta}_1, \ldots, \mathbf{x}-\boldsymbol{\eta}_m; t) = 0, \tag{4.2}$$

the incompressibility condition of the Cameron–Martin–Wiener functionals.

The pressure $P(x, t)$ in the Navier–Stokes equation may be eliminated with the aid of the incompressibility of the fluid by taking the divergence of the Navier–Stokes equation. Hence, the Navier–Stokes equation can be written†

$$\begin{aligned}\frac{\partial}{\partial t} u_i(\mathbf{x}, t) - \frac{1}{R_0} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) \\ = -\frac{\partial}{\partial x_j} u_i(\mathbf{x}, t)\, u_j(\mathbf{x}, t) - \frac{1}{4\pi} \int \frac{1}{|\mathbf{x}-\mathbf{x}'|} \frac{\partial}{\partial x_i'} \frac{\partial}{\partial x_j'} \frac{\partial}{\partial x_l'} (u_j(\mathbf{x}', t)\, u_l(\mathbf{x}', t))\, d\mathbf{x}'.\end{aligned} \tag{4.3}$$

In order to find the equations of motion of the kernels, we at first substitute the expansion of the velocity field into the Navier–Stokes equation now given by (4.3), multiply by $(1/m!) H^{(m)}_{\alpha_1 \ldots \alpha_m}(\mathbf{y}_1, \ldots, \mathbf{y}_m)$, and finally take the expectation value, to obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{R_0} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j}\right) K^{(m)}_{i\alpha_1 \ldots \alpha_m}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m; t) \\ = -\frac{\partial}{\partial x_j} \sum_{q=0}^{\infty}{}' \sum_{p=0}^{m}{}' \frac{(p+q)!\,(m-p+q)!}{m!\,q!} \mathscr{S} \int \ldots \int d\mathbf{z}_1 \ldots d\mathbf{z}_q \\ \times K^{(p+q)}_{i\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{z}_1, \ldots, \mathbf{z}_q; t) \\ \times K^{(m-p+q)}_{j\alpha_{p+1} \ldots \alpha_m \beta_1 \ldots \beta_q}(\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m, \mathbf{z}_1, \ldots, \mathbf{z}_q; t) \\ - \frac{1}{4\pi} \int d\mathbf{x}' \frac{1}{|\mathbf{x}-\mathbf{x}'|} \frac{\partial}{\partial x_i'} \frac{\partial}{\partial x_j'} \frac{\partial}{\partial x_l'} \sum_{q=0}^{\infty}{}' \sum_{p=0}^{m}{}' \frac{(p+q)!\,(m-p+q)!}{m!\,q!} \\ \times \mathscr{S} \int \ldots \int K^{(p+q)}_{j\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}(\mathbf{x}_1', \ldots, \mathbf{x}_p', \mathbf{z}_1', \ldots, \mathbf{z}_q'; t) \\ \times K^{(m-p+q)}_{l\alpha_{p+1} \ldots \alpha_m \beta_1 \ldots \beta_q}(\mathbf{x}_{p+1}', \ldots, \mathbf{x}_m', \mathbf{z}_1', \ldots, \mathbf{z}_q'; t)\, d\mathbf{z}_1' \ldots d\mathbf{z}_q',\end{aligned} \tag{4.4}$$

where the primes on the summations indicate the restrictions,

$$(p+q) > 0, \quad (m-p+q) > 0,$$

† The Navier–Stokes equation derived at (4.3) is taken to be dimensionless; the velocity is measured in units of the root-mean-square of the initial velocity and the length is measured in units of some characteristic length, hence time is measured in units of the characteristic length divided by the root-mean-square of the initial velocity. Therefore the only parameter appearing in (4.3) is R_0, which is considered the initial Reynolds number.

and where we have put

$$\mathbf{x}_1 = \mathbf{x} - \mathbf{y}_1, \quad \ldots, \quad \mathbf{x}_1' = \mathbf{x}' - \mathbf{y}_1, \quad \ldots,$$
$$\mathbf{z}_1 = \mathbf{x} - \boldsymbol{\eta}_1, \quad \ldots, \quad \mathbf{z}_1' = \mathbf{x}' - \boldsymbol{\eta}_1, \quad \ldots.$$

$\mathscr{S}$ denotes the sum of all possible different terms obtained by exchanging the variables and indices between the two sets $(\mathbf{x}_1, \alpha_1; \ldots; \mathbf{x}_p, \alpha_p)$ and $(\mathbf{x}_{p-1}, \alpha_{p-1}; \ldots; \mathbf{x}_m, \alpha_m)$. It is not difficult to see that the summation operator $\mathscr{S}$ is an operator which symmetrizes the integrals of (4.4) about their variables and indices, $(\mathbf{x}_1, \alpha_1; \ldots; \mathbf{x}_m, \alpha_m)$.

In a similar way we get, for the Burgers equation

$$\frac{\partial}{\partial t} u(x,t) + u(x,t) \frac{\partial}{\partial x} u(x,t) = \frac{1}{R_0} \frac{\partial^2}{\partial x^2} u(x,t) \tag{4.5}$$

(Burgers 1950), the following equation for the kernel of order m:

$$\left(\frac{\partial}{\partial t} - \frac{1}{R_0}\frac{\partial^2}{\partial x^2}\right) K^{(m)}(x_1, \ldots, x_m; t) = -\frac{1}{2}\frac{\partial}{\partial x} \sum_{\substack{q=0 \\ p+q>0}}^{\infty}{}' \sum_{\substack{p=0 \\ m-p+q>0}}^{m}{}' \frac{(p+q)!\,(m-p+q)!}{m!\,q!}$$
$$\times \mathscr{S} \int \ldots \int dz_1 \ldots dz_q K^{(p+q)}(x_1, \ldots, x_p, z_1, \ldots, z_q; t)$$
$$\times K^{(m-p+q)}(x_{p+1}, \ldots, x_m, z_1, \ldots, z_q; t). \tag{4.6}$$

5. Approach to the Gaussian distribution in the linear stage

In the late decay stage of turbulence, assuming the non-linear terms can be neglected in the Navier–Stokes equation, Batchelor (1953) argues semi-quantitatively that the velocity field tends to a Gaussian distribution. In this section, we give a rigorous proof using the Cameron–Martin–Wiener expansion. A part of this work has been reported already (Kahng & Siegel 1967). As a test of the Gaussianity of the velocity field, we compute the skewness and flatness factors. Our computations assume a completely arbitrary initial velocity field.

The solution of the linearized Navier–Stokes equation,

$$\frac{\partial}{\partial t} u_i(\mathbf{x}, t) - \frac{1}{R_0} \nabla^2 u_i(\mathbf{x}, t) = 0, \tag{5.1}$$

for the initial value problem is given by

$$u_i(\mathbf{x}, t) = \left(\frac{R_0}{4\pi t}\right)^{\frac{3}{2}} \int d\mathbf{z}\, u_i(\mathbf{x}+\mathbf{z}; 0) \exp\{-(R_0/4t) z^2\}. \tag{5.2}$$

Obviously the space derivatives of the velocity field,

$$u_i^{(\sigma_x, \sigma_y, \sigma_z)}(\mathbf{x}, t) = \left(\frac{\partial}{\partial x}\right)^{\sigma_x} \left(\frac{\partial}{\partial y}\right)^{\sigma_y} \left(\frac{\partial}{\partial z}\right)^{\sigma_z} u_i(\mathbf{x}, t),$$

where $\sigma_x, \sigma_y, \sigma_z$ are non-negative integers, obey the same type of equation (5.1) and have solutions similar to (5.2). The Cameron–Martin–Wiener kernels of the random variable $u_i^{(\sigma_x \sigma_y \sigma_z)}(\mathbf{x}, t)$ for the initial value problem are readily obtained by applying the orthogonality relations of Cameron–Martin–Wiener

functionals. Since the incompressiblity of the fluid allows $K^{(m)}$ to be exposed in terms of a function,

$$L^{(m)}_{n\alpha_1\ldots\alpha_m}(\mathbf{x}_1,\ldots,\mathbf{x}_m;t),$$

defined by $$K^{(m)}_{i\alpha_1\ldots\alpha_m}(\mathbf{x}_1,\ldots,\mathbf{x}_m;t)=\epsilon_{iln}\frac{\partial}{\partial x_l}L^{(m)}_{n\alpha_1\ldots\alpha_m}(\mathbf{x}_1,\ldots,\mathbf{x}_m;t), \tag{5.3}$$

we have the solution, expressed by the Fourier transform of the Cameron–Martin–Wiener kernels, as follows:

$$\begin{aligned}\tilde{K}^{(m)(\sigma_x\sigma_y\sigma_z)}_{i\alpha_1\ldots\alpha_m}(\mathbf{k}_1,\ldots,\mathbf{k}_m;t)&=(-i)^{\sigma_x+\sigma_y+\sigma_z+1}\epsilon_{iln}(k_{1x}+\ldots+k_{mx})^{\sigma_x}\\&\times(k_{1y}+\ldots+k_{my})^{\sigma_y}(k_{1z}+\ldots+k_{mz})^{\sigma_z}(k_{1l}+\ldots+k_{ml})\\&\times\tilde{L}^{(m)}_{n\alpha_1\ldots\alpha_m}(\mathbf{k}_1,\ldots,\mathbf{k}_m;0)\exp\{-(t/R_0)(\mathbf{k}_1+\ldots+\mathbf{k}_m)^2\},\end{aligned}$$

where $$\tilde{K}^{(m)(\sigma_x\sigma_y\sigma_z)}_{i\alpha_1\ldots\alpha_m}(\mathbf{k}_1,\ldots,\mathbf{k}_m;t) \tag{5.4}$$

is the Fourier transform of

$$\left(\frac{d}{dx}\right)^{\sigma_x}\left(\frac{d}{dy}\right)^{\sigma_y}\left(\frac{d}{dz}\right)^{\sigma_z}K^{(m)}_{i\alpha_1\ldots\alpha_m}(\mathbf{x}_1,\ldots,\mathbf{x}_m;t)$$

and k_{ml}, for example, means the lth component of the vector $\mathbf{k}_m$.

The computation of the second-order moments has been done essentially in § 2. Expressing them in terms of the Fourier transforms of Cameron–Martin–Wiener kernels which are the solutions of the linearized Navier–Stokes equation (5.4), and incorporating the incompressibility relation (5.3), we get

$$\begin{aligned}&\langle u_i^{(\sigma_{1x}\sigma_{1y}\sigma_{1z})}(\mathbf{x},t)\,u_j^{(\sigma_{2x}\sigma_{2y}\sigma_{2z})}(\mathbf{x},t)\rangle\\&\quad=\sum_{m=1}^{\infty}\frac{m!}{(2\pi)^{3m}}\epsilon_{iln}\epsilon_{jpq}(-i)^{\sigma_{1x}\sigma_{1y}\sigma_{1z}+1}(+i)^{\sigma_{2x}\sigma_{2y}\sigma_{2z}+1}\\&\quad\times\int\ldots\int(k_{1x}+\ldots+k_{mx})^{\sigma_{1x}+\sigma_{2x}}(k_{1y}+\ldots+k_{my})^{\sigma_{1y}+\sigma_{2y}}(k_{1z}+\ldots+k_{mz})^{\sigma_{1z}+\sigma_{2z}}\\&\quad\times(k_{1l}+\ldots+k_{ml})(k_{1p}+\ldots+k_{mp})\\&\quad\times\tilde{L}^{(m)}_{n\alpha_1\ldots\alpha_m}(\mathbf{k}_1,\ldots,\mathbf{k}_m;0)\,\tilde{L}^{(m)}_{q\alpha_1\ldots\alpha_m}(-\mathbf{k}_1,\ldots,-\mathbf{k}_m;0)\\&\quad\times\exp\{-(2t/R_0)(\mathbf{k}_1+\ldots+\mathbf{k}_m)^2\}\,d\mathbf{k}_1\ldots d\mathbf{k}_m,\end{aligned} \tag{5.5}$$

where l, n, p and q as well as $\alpha_1,\ldots,\alpha_m$ are running indices, so that the summation conventions are applied. It is easy to see that (5.5) vanishes, unless

$$\sigma_{1x}+\sigma_{2x}+\sigma_{1y}+\sigma_{2y}+\sigma_{1z}+\sigma_{2z}=\text{even integer}.$$

This is nothing but the consequence of the reflexion symmetry of the velocity field which is a part of the requirements of the isotropic field. Hence, the right-hand side of (5.5) is real, as it should be. Let us introduce a new variable such that

$$\mathbf{k}_1+\ldots+\mathbf{k}_m=\boldsymbol{\kappa}_1. \tag{5.6}$$

If we put $$\left(\frac{t}{R_0}\right)^{\frac{1}{2}}\boldsymbol{\kappa}_1=\mathbf{q}_1, \tag{5.7}$$

then, in the limit $(R_0/t)\to 0$, it is easy to see that the second-order moments decrease in the following way, to lowest order in R_0/t:

$$\begin{aligned}&\lim_{R_0/t\to 0}\langle u_i^{(\sigma_{1x}\sigma_{1y}\sigma_{1z})}(\mathbf{x},t)\,u_j^{(\sigma_{2x}\sigma_{2y}\sigma_{2z})}(\mathbf{x},t)\rangle\\&\qquad\approx\mathcal{O}((R_0/t)^{\frac{1}{2}(\sigma_{1x}+\sigma_{1y}+\sigma_{1z}+\sigma_{2x}+\sigma_{2y}+5)}).\end{aligned} \tag{5.8}$$

As a particular case, let us consider the following second-order moment,

$$\langle u_x^{(n)}(\mathbf{x},t)\, u_x^{(n)}(\mathbf{x},t)\rangle,$$

where n indicates the number of derivatives with respect to x. (x is not a running index, so no summation is applied to it.) One finds, with quite similar procedures,

$$\lim_{R_0/t\to 0} \langle u_x^{(n)}(\mathbf{x},t)\, u_x^{(n)}(\mathbf{x},t)\rangle = c_2 \frac{(2n-1)!!}{4^n}\left(\frac{R_0}{t}\right)^{n+\frac{5}{2}}, \tag{5.9}$$

where c_2 is a constant *which depends only on the initial values of the L's and, in particular, does not depend on* (R_0/t) *or* n.

As is clear from the above expressions, every kernel of the Cameron–Martin–Wiener expansion contributes to the second-order moment in the same degree. This indicates that the higher kernels do not vanish, even though (as will be shown shortly) the velocity field becomes Gaussian in the late stage. This tends to suggest that it may be possible to find a transformation of the differential space (probability space of the ideal random function) that will make the higher kernels of the Cameron–Martin–Wiener expansion decrease in proportion to the approach to Gaussianity (Siegel, Imamura & Meecham 1965).

The computations of the third-order moment and the fourth-order cumulant are straightforward if we follow the method employed in computing the second-order moment. So that we get

$$\lim_{R_0/t\to 0} \langle u_i^{(\sigma_{1x}\sigma_{1y}\sigma_{1z})}(\mathbf{x},t)\, u_j^{(\sigma_{2x}\sigma_{2y}\sigma_{2z})}(\mathbf{x},t)\, u_l^{(\sigma_{3x}\sigma_{3y}\sigma_{3z})}(\mathbf{x},t)\rangle \approx \mathcal{O}[(R_0/t)^{\frac{1}{2}\left\{\sum_{i=1}^{3}(\sigma_{ix}\sigma_{iy}\sigma_{iz})+9\right\}}] \tag{5.10}$$

and

$$\lim_{R_0/t\to 0} N_4(i,j,l,m;t) = \mathcal{O}[(R_0/t)^{\frac{1}{2}\left\{\sum_{i=1}^{4}(\sigma_{ix}+\sigma_{iy}+\sigma_{iz})+13\right\}}]. \tag{5.11}$$

A particular case of the fourth-order cumulant,

$$N_4^{(n)}(t) \equiv \langle (u_x^{(n)}(\mathbf{x},t))^4\rangle - 3(\langle (u_x^{(n)}(\mathbf{x},t))^2\rangle)^2,$$

gives the following relation,

$$\lim_{R_0/t\to 0} N_4^{(n)}(t) = c_4\left(\frac{R_0}{t}\right)^{2n+(13/2)} \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{2r}(-)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2r}{p}\cdot \times \frac{(2n-2r-1)!!\,(n+2r-s-p-1)!!\,(n+s+p-1)!!}{2^{2n+(3/2)(n+s+p)}\,3^{r-s-p}}, \tag{5.12}$$

where c_4 is a constant which is independent of n and (R_0/t).

In order to test the Gaussianity of the field, we look for the skewness and flatness factors respectively,

$$(s\cdot f)_{\sigma_x\sigma_y\sigma_z} = \frac{\langle (u_i^{(\sigma_x\sigma_y\sigma_z)}(\mathbf{x},t))^3\rangle}{(\langle (u_i^{(\sigma_x\sigma_y\sigma_z)}(\mathbf{x},t))^2\rangle)^{\frac{3}{2}}} \tag{5.13}$$

and

$$(f\cdot f)_{\sigma_x\sigma_y\sigma_z} = \frac{\langle (u_i^{(\sigma_x\sigma_y\sigma_z)}(\mathbf{x},t))^4\rangle}{(\langle (u_i^{(\sigma_x\sigma_y\sigma_z)}(\mathbf{x},t))^2\rangle)^2}. \tag{5.14}$$

From (5.10) and (5.11), we readily see that

$$\lim_{R_0/t_0\to 0} (s\cdot f)_{\sigma_x\sigma_y\sigma_z} = \mathcal{O}((R_0/t)^{\frac{3}{4}}) \tag{5.15}$$

and

$$\lim_{R_0/t\to 0} (f\cdot f)_{\sigma_x\sigma_y\sigma_z} = 3+\mathcal{O}((R_0/t)^{\frac{3}{2}}). \tag{5.16}$$

In particular, the flatness factor of the nth derivative of a component of the velocity field,

$$(f\cdot f)_n \equiv \frac{\langle (u_x^{(n)}(\mathbf{x},t))^4\rangle}{(\langle (u_x^{(n)}(\mathbf{x},t))^2\rangle)^2},$$

is given from (5.9) and (5.12) as

$$\lim_{R_0/t\to 0} (f\cdot f)_n = 3+c\cdot\left(\frac{R_0}{t}\right)^{\frac{3}{2}} \frac{4^n}{[(2n-1)!!]^2} \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{2r} (-)^{r+s} \times \binom{n}{r}\binom{n}{s}\binom{2r}{p} \frac{(2n-2r-1)!!(n+2r-s-p-1)!!(n+s+p-1)!!}{2^{\frac{3}{2}(n+s+p)}\,3^{r-s-p}} \tag{5.17}$$

where c is a constant.

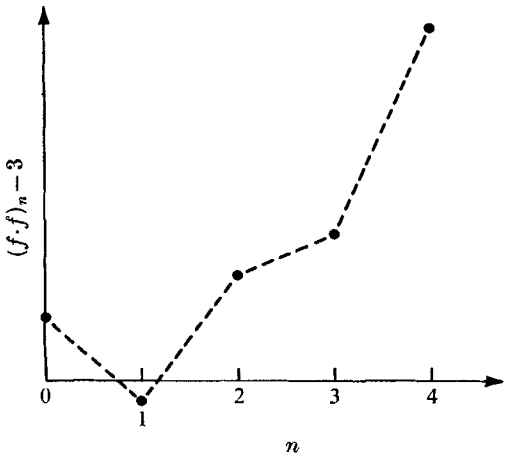

FIGURE 1. Flatness factor of velocity derivatives in the late decay stage of turbulence. Unit of vertical axis is proportional to $(R_0/t)^{\frac{3}{2}}$.

In figure 1, we plot the flatness factor minus three, on an arbitrary scale, against n. It is seen to increase rapidly with n, apart from an oscillation of period 2. Since this result is obtained from a linearized equation, for a wide variety of initial conditions, it appears that the rapid flatness factor increase with n (Batchelor 1953) observed in turbulence experiments may be purely a property of the random aspects of the process, rather than of the non-linearity of the Navier–Stokes equation.

The linearized Burgers equation is the one-dimensional case of the linearized Navier–Stokes equation. Hence, by applying the same procedures, it is possible to show that, in the Burgers model of turbulence, the skewness factor of an nth derivative of the velocity field decays as $(R_0/t)^{\frac{1}{4}}$ (if n is even, it vanishes identically) and the flatness factor minus three decays as $(R_0/t)^{\frac{1}{2}}$.

The above conclusions are derived by omitting completely the non-linear terms of the Navier–Stokes of the Burgers equation. Even if we retain the non-linear terms as a perturbation, the Gaussian trend of the velocity field can be proved. However, the n dependence of the skewness and flatness factors would depend on the initial velocity field until the linear stage was reached. Results of perturbation theory for the Burgers model of turbulence will be obtained in §6.6.

6. Burgers' model of turbulence in the late decay stage

6.1. *Use of the Hopf–Cole solution*

In this section the method of the Cameron–Martin–Wiener expansion is applied to the investigation of the late decay problem of Burgers' model of turbulence, paying particular attention to the non-linear aspects of the process.

If we take the time origin late enough, the initial Reynolds number R_0 is very small. Therefore R_0 ought to be a good expansion parameter for any quantity in this stage. The solution of the Burgers equation for the initial value problem (Hopf 1950; Cole 1951),

$$u(x,t) = -\frac{2}{R_0}\frac{\partial}{\partial x}\log\theta(x,t), \tag{6.1}$$

where
$$\theta(x,t) = \left(\frac{R_0}{4\pi t}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left(-\frac{R_0}{4t}z^2\right)\exp\left[-\frac{R_0}{2}\int_{x_0}^{x+z}u(s,0)\,ds\right], \tag{6.1a}$$

can be expanded in powers of R_0 as

$$u(x,t) = \sum_{n=0}^{\infty} u_{(n)}(x,t), \tag{6.2}$$

where

$$u_{(0)}(x,t) = \frac{\partial}{\partial x}\left(\frac{R_0}{4\pi t}\right)^{\frac{1}{2}}\int\exp\left[-(R_0/4t)z^2\right]\left(\int_{x_0}^{x+z}u(s,0)\,ds\right)dz, \tag{6.3a}$$

$$u_{(1)}(x,t) = -\frac{R_0}{4}\frac{\partial}{\partial x}\left(\frac{R_0}{4\pi t}\right)^{\frac{1}{2}}\int\exp\left[-(R_0/4t)z^2\right]\left(\int_{x_0}^{x+z}u(s;0)\,ds\right)^2 dz$$
$$+\frac{R_0}{4}\frac{\partial}{\partial x}\left[\left(\frac{R_0}{4\pi t}\right)^{\frac{1}{2}}\int\exp\left[-(R_0/4t)\,z^2\right]\left(\int_{x_0}^{x+z}u(s;0)\,ds\right)dz\right]^2, \tag{6.3b}$$

$$u_{(2)}(x,t) = \ldots.\dagger \tag{6.3c}$$

One can easily show that the expansion (6.2) satisfies the perturbed Burgers equation taking the non-linear term as a perturbation.

6.2. *Choice of the initial condition*

Because the Hopf–Cole solution (and thus the expansion developed in §6.2) is the solution of the initial value problem, the choice of the initial velocity field, which should be realizable, is all-important. But we do not know how the velocity field has developed from the early stage. The most interesting way to

† Dots on the right-hand side of an equation indicate that a lengthy expression has been omitted. A copy of the full text may be obtained from either the senior author (A.S.) or the editorial office of the *Journal*.

proceed seemed to be to take, as representative cases, the following two initial velocity fields for the discussion of this section:

$$(a) \quad u(x,t) = \frac{\partial}{\partial x}\int L^{(1)}(x-\eta;0)\,H^{(1)}(\eta)\,d\eta, \tag{6.4}$$

and

$$(b) \quad u(x,t) = \frac{\partial}{\partial x}\int L^{(1)}(x-\eta;0)\,H^{(1)}(\eta)\,d\eta + \frac{\partial}{\partial x}\iint L^{(2)}(x-\eta_1, x-\eta_2;0)\,H^{(2)}(\eta_1,\eta_2)\,d\eta_1 d\eta_2, \tag{6.5}$$

where the function $L^{(m)}(x_1,\ldots,x_m;t)$ is defined by

$$K^{(m)}(x_1,\ldots,x_m;t) = \frac{\partial}{\partial x}L^{(m)}(x_1,\ldots,x_m;t) \tag{6.6a}$$

or

$$\tilde{K}^{(m)}(k_1,\ldots,k_m;t) = (-i)\,(k_1+\ldots+k_m)\,\tilde{L}^{(m)}(k_1,\ldots,k_m;t). \tag{6.6b}$$

These are, of course, 'initial' fields only relative to the late decay stage. With these choices we can evaluate the behaviour of a pure Gaussian field and the effect of a non-Gaussian addition to it. The introduction of the function $L^{(m)}(k_1,\ldots,k_m;t)$ is motivated by the fact that properties (Siegel & Kahng 1969; Kahng & Siegel 1968)

$$\tilde{L}^{(m)}(k_1,\ldots,k_m;t) = \tilde{L}^{(m)}(-k_1,\ldots,-k_m;t) \tag{6.7}$$

and

$$\tilde{L}^{(m)}(0,\ldots,0;t) \not\equiv 0. \tag{6.8}$$

The choice of this form for the kernels is in accordance with the following, as explained in detail in the paper by Siegel & Kahng cited above: It is shown there that the odd-numbered kernels must be either even or odd functions in the isotropic field. It is then reasoned that if the resolution of this choice is made according to the criteria that the Burgers model should be set up to parallel incompressible three-dimensional turbulence as closely as possible and to have a Reynolds number which approaches zero in late decay, and that a quasi-linear stage should even exist, the decision must be in favour of odd functions; and these functions, in view of the circumstances pointed out in the article, can be written in the form given.

It is noteworthy that certain unpleasant potential features of the Cameron–Martin–Wiener expansion in the Burgers model are eliminated by the choice of odd parity for odd-numbered kernels. One such unpleasant feature, pointed out in the previous reference, is that the second-order kernel when the odd-numbered kernels are even functions has to be larger than the first-order one, which means that if the field is close to Gaussian, a large number of higher-order kernels have to be carried along in the calculation in order to cancel the effects of the non-Gaussian second-order kernel; thus truncation to a few terms of the expansion becomes impossible. Another is that, without the restriction to odd parity, the solution to the Burgers equation in the quasi-linear stage gives rise to features that are non-uniform in time; whereas, as will appear in the present paper, the odd-parity restriction leads to a solution having a self-similarity property in time.

On the other hand, Professor A. Newell of the University of California at Los Angeles has expressed to us the opinion that work of Lange (1969) may imply the invalidation of one of the assumptions on which the above argument is based, namely, that of linearization in late decay. In particular, this would not permit the deduction of the $t^{-\frac{1}{2}}$ decay law found in Siegel & Kahng just after their (38). Until it is shown that the argumentation of Siegel & Kahng either survives this potential criticism or can be reconstituted with some perhaps altered assumption, it is not possible to exclude the even-parity choice completely. However, as shown by Siegel & Kahng, the requirement that the odd-numbered kernels must be *either all even or all odd, but not mixed* is the most general way to satisfy the isotropy condition that it is possible to formulate clearly thus far. Hence on the ensuing pages we study at the very least one of at most two mutually exclusive possibilities, at least as far as present understanding goes.

The choice (*a*), (6.4), the Gaussian initial velocity field, is of special interest, not merely because of its mathematical simplicity, but because of the Gaussian tendency of the velocity field in the late decay stage, which was generally proved in §5. However, experiments and theory both indicate the importance of the departure from Gaussianity. It is for this reason that the non-Gaussian velocity case (*b*) is also taken up. This may help to understand in what direction we will need to correct the oversimplified conclusions to which case (*a*) must necessarily lead.

6.3. *Solutions for the Cameron–Martin–Wiener kernels*

Given the expanded form of the Hopf–Cole solution (6.2), the time-dependent kernels are obtained by substituting the initial conditions, multiplying $(I/m!)H^{(m)}(y_1, \ldots, y_m)$, and taking the expectation value utilizing the orthogonality relation of the functionals. We obtain the following, in terms of Fourier transforms:

(i) *Gaussian initial velocity field*:

$$\tilde{K}^{(1)}(k_1, t) = \ldots, \tag{6.9}$$

$$\tilde{K}^{(2)}(k_1, k_2; t) = \ldots, \tag{6.10}$$

$$\tilde{K}^{(3)}(k_1, k_2, k_3; t) = \ldots, \tag{6.11}$$

$$\tilde{K}^{(m)}(k_1, \ldots, k_m; t) = \ldots. \tag{6.12}$$

(ii) *Non-Gaussian initial velocity field*:

$$\tilde{K}^{(1)}(k_1; t) = \ldots, \tag{6.13}$$

$$\tilde{K}^{(2)}(k_1, k_2; t) = \ldots, \tag{6.14}$$

$$\tilde{K}^{(3)}(k_1, k_2, k_3; t) = \ldots, \tag{6.15}$$

$$\tilde{K}^{(4)}(k_1, k_2, k_3, k_4; t) = \ldots, \tag{6.16}$$

$$\tilde{K}^{(5)}(k_1, k_2, k_3, k_4, k_5; t) = \ldots, \tag{6.17}$$

$$\vdots$$

Since the late stage is characterized by the smallness of the Reynolds number, the order of smallness of each term will be measured by its power of R_0. $\tilde{K}^{(m)}(k_1, \ldots, k_m; t)$ in this expansion consists of an infinite series, whose successive terms become smaller and smaller. However, the higher order kernels are not necessarily small compared with the lower order kernels. Rather the smallness of the higher-order kernels is highly dependent on the choice of initial conditions.

It should be noted that, in the explicit expression of the Cameron–Martin–Wiener kernels, no trace remains of the arbitrary constant x_0 which appeared in the Hopf–Cole solution (6.1).

The approximation of the Cameron–Martin–Wiener kernels presented above can also be examined directly from the equations of motion of the kernels. In fact, it can be shown that the solutions we have obtained satisfy the perturbed equations of motion.

Our present result completely agrees with that of Siegel, Imamura & Meecham (1965), if we choose the same initial velocity field. In the present approach, however, the smallness of the higher kernels comes out as a natural consequence of the late stage.

6.4. *Non-linear process in the late decay stage*

The counterpart in the Burgers model of the Kármán–Howarth (1938) equation is

$$\frac{\partial}{\partial t} Q(r,t) = \frac{2}{R_0} \frac{\partial^2}{\partial r^2} Q(r,t) + \frac{\partial}{\partial r} S(r,t), \tag{6.18}$$

where
$$Q(r,t) = \langle u(x;t)\, u(x+r;t) \rangle \tag{6.18a}$$

and
$$S(r,t) = \langle u(x,t) u(x,t) u(x+r;t) \rangle. \tag{6.18b}$$

We here propose to utilize the general expressions of the preceding subsections in order to investigate non-linear effects in the late decay state in some detail. In doing so, we are of course motivated by the hope that this region, in which non-linear effects are relatively easy to study, may provide insights toward the understanding of the more strongly non-linear region. The results to be given are of perturbative type. They differ from previous perturbative analyses in possessing the peculiar advantages of the Cameron–Martin–Wiener expansion, principally that of giving an energy spectrum which is always inherently positive.

It will be convenient to introduce an auxiliary function $\mathscr{R}(r,t)$ defined by

$$\mathscr{R}(r,t) = Q(r,t) - Q'(r,t), \tag{6.19}$$

where $Q'(r,t)$ satisfies the linear equation

$$\frac{\partial}{\partial t} Q'(r,t) = \frac{2}{R_0} \frac{\partial^2}{\partial r^2} Q'(r,t). \tag{6.20}$$

Furthermore, $\mathscr{R}(r,t)$ is taken so that $(2/R_0)(\partial^2/\partial r^2)\mathscr{R}(r,t)$ and $(\partial/\partial r) S(r,t)$ are of the same order in the powers of the expansion parameter; this determines $\mathscr{R}$ and makes all terms in (6.21), below, of the same order. Because of (6.20), $\mathscr{R}(r,t)$ satisfies the equation,

$$\frac{\partial}{\partial t} \mathscr{R}(r,t) = \frac{2}{R_0} \frac{\partial^2}{\partial r^2} \mathscr{R}(r,t) + \frac{\partial}{\partial r} S(r,t), \tag{6.21}$$

which is of exactly the same form as the Kármán–Howarth equation of the Burgers model.

We have divided the correlation function into two parts. One of them is independent of the non-linear process (at the instant itself), and the other is associated with the non-linear process. Although one cannot expect to achieve a clean separation of linear from non-linear effects, we may suppose that any effect involving $\mathscr{R}(r,t)$ must be quite essentially of non-linear nature.

The same procedure can be carried out for the energy spectrum density. We define

$$\mathscr{E}(k,t) = E(k,t) - E'(k,t), \tag{6.22}$$

so that

$$\frac{\partial}{\partial t}E'(k,t) + \frac{2}{R_0}k^2E'(k,t) = 0 \tag{6.23}$$

and

$$\frac{\partial}{\partial t}\mathscr{E}(k,t) + \frac{2}{R_0}k^2\mathscr{E}(k,0) + T(k,t) = 0, \tag{6.24}$$

in which we impose the condition that $(\partial/R_0)k^2\mathscr{E}(k,t)$ and $T(k,t)$ are of the same order. $\mathscr{E}(k,t)$ is the Fourier transform of $\mathscr{R}(r,t)$. Although the energy spectrum density $E(k,t)$ is positive, $\mathscr{E}(k,t)$ is not necessarily so. $\mathscr{E}(k,t)$ is to be understood as the deviation of the energy spectrum density from a certain definite form, $E'(k,t)$, which is the main part of the energy spectrum density in the late decay stage. We shall call the latter the 'equilibrium energy spectrum density', because it will be seen to be expressible in terms of a self-similar functional form independent of time (the time dependence taking place through the variables and through the multiplicative coefficients).

6.5. *The spectral analysis*

(i) *Gaussian initial velocity field.* The energy spectrum density in the late stage obtained from the kernels (6.9)–(6.12) is

$$E(k,t) = \ldots. \tag{6.25}$$

The successive terms are seen to be of smaller and smaller order in R_0^2. From this energy spectrum density, the equilibrium energy spectrum density $E'(k,t)$, its deviation $\mathscr{E}(k,t)$, and the energy transfer term $T(k,t)$ are obtained as

$$E'(k,t) = \ldots, \tag{6.26}$$

$$\mathscr{E}(k,t) = \ldots, \tag{6.27}$$

$$T(k,t) = \ldots. \tag{6.28}$$

It should be emphasized at this point that $E'(k,t)$ which behaves linearly from the point of view of the Kármán–Howarth equation, nonetheless contains terms of non-linear origin from the point of view of the velocity function (e.g. terms in $E'(k,t)$ of fourth order in $\tilde{L}^{(1)}$). Thus the present theory is perturbative from the point of view of the Kármán–Howarth equation, but better than perturbative from that of the equation of motion of the velocity field, since even its zero order contains non-linear elements.

By direct calculation, one sees that

$$\int_0^\infty T(k,t)\,dk = 0, \tag{6.29}$$

which shows that the present approximation satisfies the energy conservation law for the non-linear term.

Let us now look at the Loitsianskii parameter of the Burgers model of turbulence as the coefficient of k^2 when the energy spectrum density is expanded in powers of k (Siegel & Kahng 1969; Kahng & Siegel 1968). In the present approximation, it is given by

$$\mathscr{L}(t) = \left.\frac{E'(k,t)}{k^2}\right|_{k=0} + \left.\frac{\mathscr{E}(k,t)}{k^2}\right|_{k=0}, \tag{6.30}$$

where

$$\left.\frac{E'(k,t)}{k^2}\right|_{k=0} = (\tilde{L}^{(1)}(0,0))^2 + \frac{R_0^2}{16\pi}\int dk'(\tilde{L}^{(1)}(k';0)\,\tilde{L}^{(1)}(-k';0))^2 + \mathcal{O}(R_0^4)$$

$$= \text{constant}, \tag{6.30a}$$

$$\left.\frac{\mathscr{E}(k,t)}{k^2}\right|_{k=0} = \frac{R_0^2}{16\pi}\int dk'(\tilde{L}^{(1)}(k';0)\,\tilde{L}^{(1)}(-k';0))^2$$

$$\times (\exp[-(4t/R_0)\,k'^2] - 2\exp[-(2t/R_0)\,k'^2]). \tag{6.30b}$$

This Loitsianskii parameter is not constant in time, unlike that of three-dimensional real turbulence. However, since the contribution to the Loitsianskii parameter from $\mathscr{E}(k,t)$ is small, we can say that the Loitsianskii parameter does tend toward a constant in the late decay stage. The constancy of the

$$\left.\frac{E'(k,t)}{k^2}\right|_{k=0}$$

part of the Loitsianskii parameter of the Burgers model of turbulence and of the Loitsianskii parameter of real turbulence all have this in common: these are parameters of the energy spectrum to which the non-linear term of the Kármán–Howarth equation does not contribute, hence embody a purely linear aspect of the flow.

A detailed study of the energy transfer process can be done by examining the energy tranfer term $T(k,t)$ directly, and by comparing it with $\mathscr{E}(k,t)$. According to (6.24), the rate of change of the energy spectrum deviation is minus the sum of the viscous dissipation of the spectrum deviation and the energy transfer,

$$\frac{\partial}{\partial t}\mathscr{E}(k,t) = -\frac{2}{R_0}k^2\mathscr{E}(k,t) - T(k,t).$$

According to this equation the viscous dissipation of the spectrum deviation tends to make the deviation of the energy spectrum density decrease in magnitude. However, if the sign of $T(k,t)$ is opposite to that of $\mathscr{E}(k,t)$, and if $T(k,t)$ is large enough, the energy transfer process may make the energy spectrum deviation, in some wave region, increase.

For the sake of definiteness, let us consider the case of small (R_0/t). Putting

$$q = \left(\frac{t}{R_0}\right)^{\frac{1}{2}}k, \tag{6.31}$$

and neglecting higher-order terms in R_0/t, we get

$$E'\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right) \approx \frac{R_0}{t}(\tilde{L}^{(1)}(0,0))^2 q^2 e^{-2q^2} + \frac{R_0^2}{16\pi}\frac{R_0}{t}\left[\int dk'(\tilde{L}^{(1)}(k';0))^4\right] q^2 e^{-2p^2}, \quad (6.32)$$

$$\mathscr{E}\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right) \approx \frac{R_0^2}{4\sqrt{2\pi}}\left(\frac{R_0}{t}\right)^{\frac{3}{2}}(\tilde{L}^{(1)}(0,0))^4 q^2\left(\frac{\sqrt{2}}{8}e^{-q^2} - \tfrac{3}{2}e^{-(\frac{3}{2})q^2} + e^{-2q^2}\right), \quad (6.33)$$

$$T\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right) \approx \frac{R_0}{8\sqrt{(2\pi)}}\left(\frac{R_0}{t}\right)^{\frac{5}{2}}(\tilde{L}^{(1)}(0,0))^4 \times q^2\left(\frac{\sqrt{2}}{8}e^{-q^2} - \frac{\sqrt{2}}{4}q^2e^{-q^2} - \tfrac{3}{2}e^{-(\frac{3}{2})q^2} + \tfrac{3}{2}q^2e^{-(\frac{3}{2})q^2} + e^{-2q^2}\right). \quad (6.34)$$

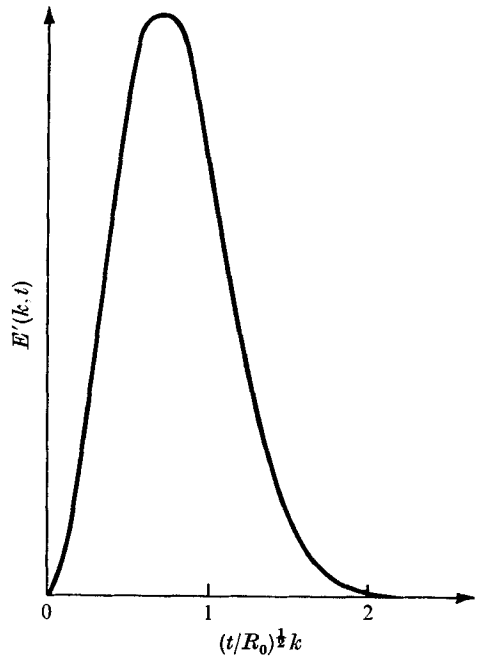

FIGURE 2. Equilibrium form of energy spectrum in the late decay stage. Unit of vertical axis is proportional to (R_0/t).

We see that the introduction of the variable q has separated two forms of time variation in $E'(k,t)$, $\mathscr{E}(k,t)$ and $T(k,t)$: (1) A shift of the wave-number abscissa values, without change in form of the functions, towards the origin at a rate proportional to (t/R_0); and (2) a reduction in amplitude by the respective factors (R_0/t), $(R_0/t)^{\frac{3}{2}}$, $(R_0/t)^{\frac{5}{2}}$, respectively.

The fact that $T(k,t)$ decays faster than $(2/R_0)k^2E'(k,t)$ is a manifestation of internal consistency in these results, since the assumed Gaussianity of the field is supposed to be a consequence of an approach to linearity with time, and $T(k,t)$ is the non-linear term.

In the late decay stage, the universal forms we have found by the use of q as variable show that any characteristic wave-number, if we assign one, varies

in proportion to $(R_0/t)^{\frac{1}{2}}$, while the scale of turbulence in this stage increases in proportion to $(t/R_0)^{\frac{1}{2}}$.

For further understanding of the non-linear aspects of the process, let us consider the ratio,

$$\gamma(k,t) = \frac{\mathscr{E}(k,t)}{E(k,t)}, \tag{6.35}$$

which measures the energy spectrum deviation per unit energy spectrum density. In this late stage, it has the form,

$$\gamma(k,t) \approx \frac{R_0^2}{4\sqrt{2\pi}} \left(\frac{R_0}{t}\right)^{\frac{1}{2}} (\tilde{L}^{(1)}(0,0))^2 \left(1 - \tfrac{3}{2} e^{(\frac{1}{2})q^2} + \frac{\sqrt{2}}{8} e^{q^2}\right). \tag{6.36}$$

The behaviour of $\gamma(k,t)$ is illustrated in figure 5. The diagram shows that the magnitude of $\gamma(k,t)$ is generally increasing for large wave-number. This shows

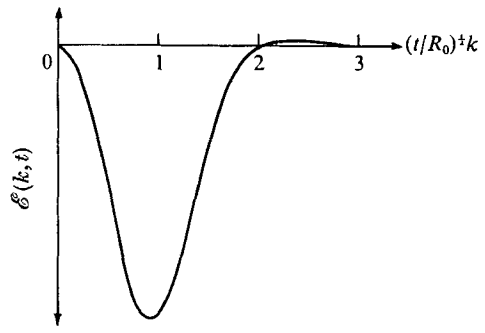

FIGURE 3

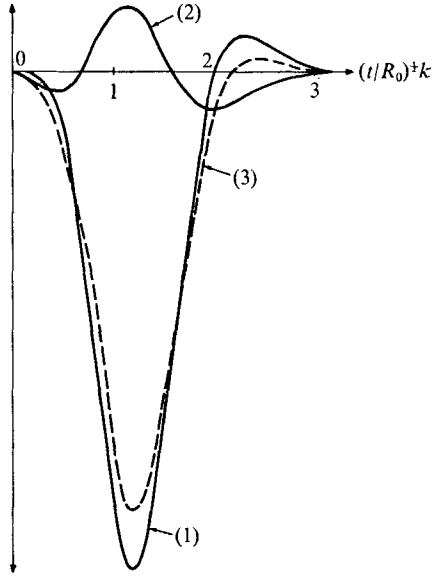

FIGURE 4

FIGURE 3. Energy spectrum deviation (Gaussian initial case). Unit of vertical axis is proportional to $R_0^2(R_0/t)^{\frac{3}{2}}$.

FIGURE 4. (1) Viscous dissipation. (2) Energy transfer term. (3) Minus the rate of change of the energy spectrum deviation (Gaussian initial case). Unit of vertical axis is proportional to $R_0(R_0/t)^{\frac{5}{2}}$.

that the non-linear aspects of the motion of large eddy size fade away faster than those of the motion of small eddy size. (Naturally, the expression (6.36) ceases to be valid when q becomes so large that $|\gamma|$ is no longer a small quantity.)

By examining the rate of change of the energy spectrum deviation per unit energy spectrum deviation, it is also possible to show that the motion of small eddy size approaches the equilibrium state more rapidly than that of large eddy size.

Now let us examine the energy transfer process itself, through the energy transfer term $T(k,t)$ which is illustrated in figure 4. In the large wave-number

region, which contains only a small amount of energy, $T(k,t)$ is shown to be negative. This agrees with the general belief that energy transfers from the large to the small eddies. In a limited region of small k, we see that $T(k,t)$ again becomes negative; which means that some amount of energy flows into this region of small k. It is seen that this phenomenon is tightly related with the negativeness of $\mathscr{E}(k,t)$ in this region of small k.

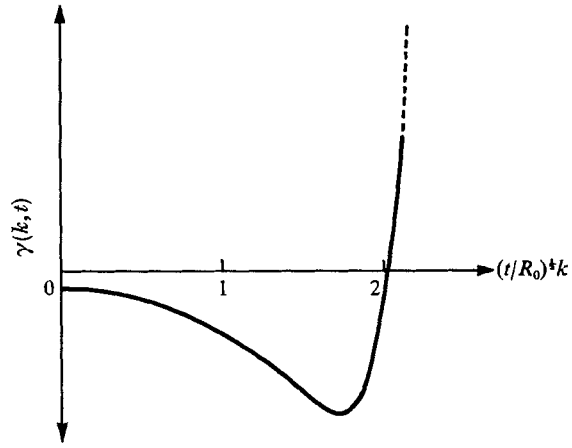

FIGURE 5. Ratio of energy spectrum deviation to energy spectrum density (Gaussian initial case). Unit of vertical axis is proportional to $R_0^2(R_0/t)^{\frac{1}{2}}$.

(ii) *Non-Gaussian initial velocity field.* The analysis of the case of non-Gaussian initial velocity field runs parallel to that of the Gaussian case. Therefore, all duplicating explanations will be omitted in the following discussion.

In this problem, we use the solutions given by (6.13)–(6.17), where the contribution of $L^{(2)}(\ldots)$ is assumed of the same order as that of $L^{(1)}(\ldots)$. The energy spectrum density is given by

$$E(k,t) = \ldots. \tag{6.37}$$

The equilibrium energy spectrum density $E'(k,t)$ and the energy spectrum deviation $\mathscr{E}(k,t)$ are given by

$$E'\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right) \approx \ldots \tag{6.38}$$

and

$$\mathscr{E}\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right) \approx \ldots, \tag{6.39}$$

where higher-order terms in (R_0/t) are neglected.

As in the Gaussian case, we can show that the Loitsianskii parameter is not constant in time but does approach a constant.

The energy transfer term $T(k,t)$ is given by

$$T\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right) \approx \ldots. \tag{6.40}$$

This expression shows that

$$\int_0^\infty T\left(\left(\frac{R_0}{t}\right)^{\frac{1}{2}}q,t\right)dq = 0, \tag{6.41}$$

which again confirms the energy conservation by the non-linear term.

In figure 6, the energy spectrum deviation $\mathscr{E}(k,t)$ is plotted. This shows that $\mathscr{E}(k,t)$ is positive everywhere, in contrast to the Gaussian case.

Let us consider the following ratio:

$$\gamma(k,t) = \frac{\mathscr{E}(k,t)}{E(k,t)} \approx c\left(\frac{R_0}{t}\right)^{\frac{1}{2}} e^{(\frac{1}{2})q^2}, \tag{6.42}$$

where

$$e = \frac{\frac{R_0}{2\sqrt{(2\pi)}}\left\{3(\tilde{L}^{(1)}(0,0))^2(\tilde{L}^{(2)}(0,0,0)) + \frac{2}{\pi}\int dk'(\tilde{L}^{(2)}(k',-k';0))^3\right\}}{(\tilde{L}^{(1)}(0,0))^2 + \frac{1}{\pi}\int dk'(\tilde{L}^{(2)}(k',-k';0))^2}.$$

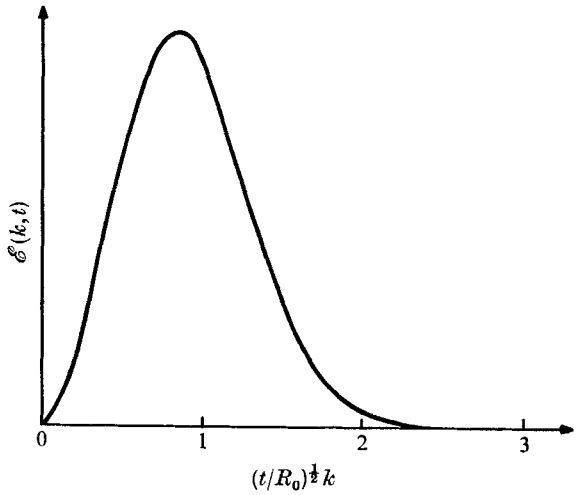

FIGURE 6. Energy spectrum deviation (non-Gaussian initial case). Unit of vertical axis is proportional to $R_0(R_0/t)^{\frac{3}{2}}$.

The curve of $\gamma(k,t)$ against q is given by figure 8. As in the Gaussian case, the magnitude of $\gamma(k,t)$ is increasing for large wave-numbers, which means again that the motion of small eddy size is less settled than that of large.

In figure 7, the curves of $-(\partial/\partial t)\mathscr{E}(k,t)$, $(2/R_0)k^2\mathscr{E}(k,t)$ and $T(k,t)$ are plotted against q. In all of wave-number space, $-(\partial/\partial t)\mathscr{E}(k,t)$ is positive, so that the magnitude of the energy spectrum deviation decays everywhere.

The curve of the energy transfer term shows that energy flows consistently from the large to the small eddies. Unlike the region of the purely Gaussian initial case, energy never flows into the region of very small k. We see that it is the positiveness of $\mathscr{E}(k,t)$ near $k = 0$ that avoids this feature.

(iii) *Comparison of Gaussian and non-Gaussian cases.* In almost all of the energy-containing region of wave-number space, the energy spectrum deviation $\mathscr{E}(k,t)$ for the Gaussian initial case and for the non-Gaussian case have opposite signs. Of course, in both cases, the process occurs in such a way as to reduce the absolute value of $\mathscr{E}(k,t)$. Both cases also show that the motion of small eddy size is less settled than that of large eddy size, but the former approaches equilibrium more rapidly than the latter.

It is the negativeness of $\mathscr{E}(k,t)$ in the region of small k that induces the negative energy flow in that region. By changing the magnitude of the initial value of $L^{(2)}(\ldots;0)$, however, the region of presumably anomalous negative energy flow at small wave-numbers can be reduced. This emphasizes the importance of the choice of the 'initial' velocity field ('initial' meaning relative to the late decay period). It should be noted that the possibility of negative energy flow in the region of small k is a characteristic of Burgers model, but not necessarily of real turbulence.

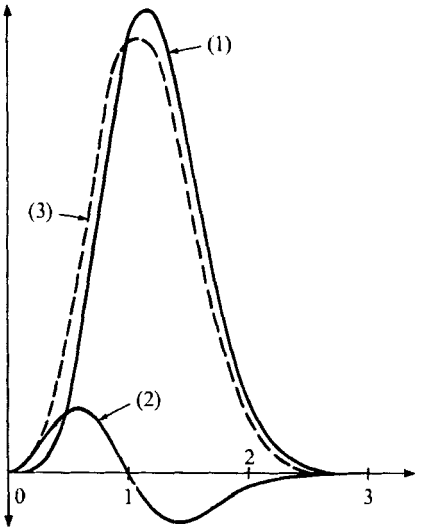

FIGURE 7

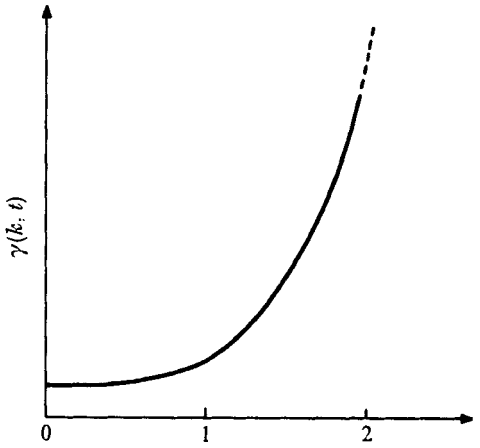

FIGURE 8

FIGURE 7. (1) Viscous dissipation. (2) Energy transfer term. (3) Minus the rate of change of the energy spectrum deviation (non-Gaussian initial case). Unit of vertical axis is proportional to $(R_0/t)^{\frac{5}{2}}$.

FIGURE 8. Ratio of energy spectrum to energy spectrum density (non-Gaussian initial case). Unit of vertical axis is proportional to $R_0(R_0/t)^{\frac{1}{2}}$.

6.6. *The velocity distribution*

A test of Gaussianity of the velocity field can be made by examining the skewness and flatness factors:

$$(s\cdot f)_n = \frac{\langle (u^{(n)}(x,t))^3\rangle}{[\langle (u^{(n)}(x,t))^2\rangle]^{\frac{3}{2}}} \tag{6.43}$$

and

$$(f\cdot f)_n = \frac{\langle (u^{(n)}(x,t))^4\rangle}{[\langle (u^{(n)}(x,t))^2\rangle]^2} = 3+\frac{N_4^{(n)}(t)}{[\langle (u^{(n)}(x,t))^2\rangle]^2}, \tag{6.44}$$

where $u^{(n)}(x,t)$ means $(d^n/dx^n)u(x,t)$ and $N_4^{(n)}(t)$ is the fourth-order cumulant of the field of $u^{(n)}(x,t)$, which is given by

$$N_4^{(n)}(t) = \langle (u^{(n)}(x,t))^4\rangle - 3[\langle (u^{(n)}(x,t))^2\rangle]^2. \tag{6.45}$$

The expressions of the second- and third-order moments and the fourth-order cumulant in terms of the Cameron–Martin–Wiener kernels are essentially the same as §2, but all of the functions and variables should be replaced by 'scalars', because the velocity field at present is one-dimensional.

(i) *Gaussian initial velocity field.* Substituting the solutions, (6.9)–(6.12), and neglecting higher-order terms of (R_0/t), we get the following:
The second-order moment is given by

$$\langle (u^{(n)}(x,t))^2 \rangle \approx \ldots . \tag{6.46}$$

The third-order moment is given by

$$\langle (u^{(n)}(x,t))^3 \rangle \approx \ldots, \tag{6.47}$$

where n has to be an odd number if there is to be a non-vanishing third-order moment, because, if n is even, this moment vanishes identically by reflexion symmetry.

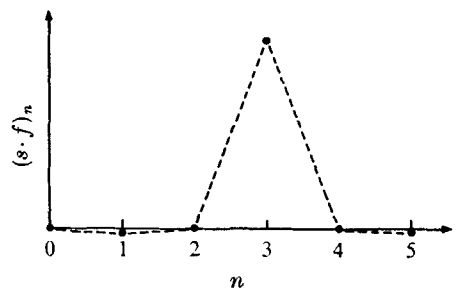

FIGURE 9

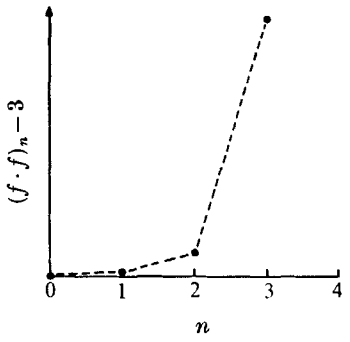

FIGURE 10

FIGURE 9. Skewness factor of velocity derivatives (Gaussian initial case). Unit of vertical axis is proportional to $R_0(R_0/t)^{\frac{1}{2}}$.

FIGURE 10. Flatness factor minus three of velocity derivatives (Gaussian initial case). Unit of vertical axis is proportional to $R_0^2(R_0/t)^{\frac{1}{2}}$.

The fourth-order cumulant is given by

$$N_4^{(n)}(t) \approx \ldots . \tag{6.48}$$

From these expressions, the skewness factor is given by

$$(s \cdot f)_n = \frac{\text{eq. (6.47)}}{(\text{eq. (6.46)})^{\frac{3}{2}}}, \tag{6.49}$$

by which we easily see that it decays as $(R_0/t)^{\frac{1}{2}}$. The dependence on n is plotted in figure 9. The flatness factor is now given by

$$(f \cdot f)_n = 3 + \frac{\text{eq. (6.48)}}{(\text{eq. (6.46)})^2}. \tag{6.50}$$

This shows that the flatness factor minus three decays as $(R_0/t)^{\frac{1}{2}}$. The dependence on n is plotted in figure 10 which shows a rapid increase with n.

(ii) *Non-Gaussian initial velocity field.* In this case, we use the solutions obtained at (6.13)–(6.17). Then the second-order moment is given by

$$\langle (u^{(n)}(x,t))^2 \rangle \approx \ldots . \tag{6.51}$$

The third-order moment is given by

$$\langle (u^{(n)}(x,t))^3 \rangle \approx \ldots,$$

$$= 0 \text{ if } n \text{ is even.} \tag{6.52}$$

The fourth-order cumulant is given by

$$N^{(n)}(t) \approx \ldots. \tag{6.35}$$

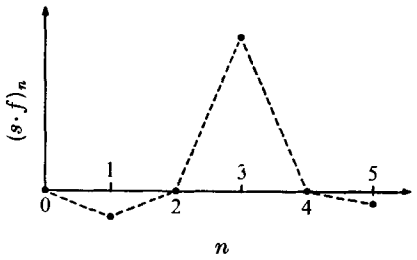

FIGURE 11

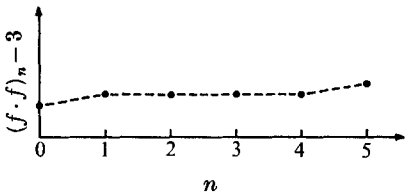

FIGURE 12

FIGURE 11. Skewness factor of velocity derivatives (non-Gaussian initial case). Unit of vertical axis is proportional to $(R_0/t)^{\frac{1}{2}}$.

FIGURE 12. Flatness factor of velocity derivatives (non-Gaussian initial case). Unit of vertical axis is proportional to $(R_0/t)^{\frac{1}{2}}$.

Therefore, the skewness and flatness factors are obtained from

$$(s\cdot f)_n = \frac{\text{eq. (6.52)}}{(\text{eq. (6.51)})^{\frac{3}{2}}} \tag{6.54}$$

and

$$(f\cdot f)_n = 3 + \frac{\text{eq. (6.53)}}{(\text{eq. (6.51)})^2}, \tag{6.55}$$

by which we see that, as in the Gaussian initial case, the skewness factor decays as $(R_0/t)^{\frac{1}{2}}$ and the flatness factor minus three decays as $(R_0/t)^{\frac{1}{2}}$. The n dependence of both of them are plotted in figures 11 and 12.

(iii) *Comparison of Gaussian and non-Gaussian cases.* A remarkable difference between Gaussian and non-Gaussian cases is seen for the dependence on n of the (flatness factor -3) for the field of velocity derivatives; in contrast to the Gaussian initial case (i), the flatness factor for the non-Gaussian initial case (ii) does not increase with n, at least not for the first few values of n. This is further evidence of the strong influence of initial conditions on the behaviour in late decay stage, as previously emphasized at the end of the preceding section and also by Siegel, Imamura & Meecham (1965). Because of this, the amount of information on the 'universal' characteristics of turbulence which can be obtained from the analysis of the quasi-linear stage, at least directly, seems highly restricted.

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